Name: ______________________ GTID: ______________________

- Fill out your name and Georgia Tech ID number.
- This quiz contains 6 pages. Please make sure no page is missing.
- The grading will be done on the scanned images of your test. Please write clearly and legibly, and outline all the details.
- Answer the questions in the spaces provided. We will scan the front sides only by default. If you run out of room for an answer, continue on the back of the page and notify the TA when handing in.
- Please write detailed solutions including all steps and computations.
- The duration of the quiz is 30 minutes.

Good luck!

1. (25 points) Find the convolution[1] of $f(t) = t$ and $g(t) = e^{-t}$.

Solution: We use the formula for the convolution and compute

$$(f \star g)(t) = \int_0^t f(t-s)g(s)ds = \int_0^t (t-s)e^{-s}ds = t + e^{-t} - 1,$$

where the last step follows by integration by parts.
Alternative: If you remember the Laplace transforms of the functions in question, we get

$$\mathcal{L}\{f \star g\}(s) = \mathcal{L}\{f\}(s)\mathcal{L}\{g\}(s) = \frac{1}{s^2}\frac{1}{s+1} = \frac{1}{s^2} + \frac{1}{s+1} - \frac{1}{s},$$

where we computed partial fractions in the last step. Undoing the Laplace transform yields

$$(f \star g)(t) = \mathcal{L}^{-1}\left(\mathcal{L}\{f \star g\}\right) = t + e^{-t} - 1.$$

[1] Note that the relevant functions are defined on $(0, \infty)$.

2. (25 points) Determine the critical points of the system

$$\frac{dx}{dt} = x(1-x-3y) \text{ and } \frac{dy}{dt} = (y+1)\left(\frac{x+2}{3}\right).$$

Solution: The critical points are found by solving the equations

$$x(1-x-3y) = 0 \text{ and } (y+1)\left(\frac{x+2}{3}\right) = 0.$$

The first equation can be satisfied by either choosing $x = 0$ or $1-x-3y = 0$ Substituting $x = 0$ into the second equation yields the critical point $(0, -1)$. Substituting $1 - x - 3y = 0$, or equivalently $x = 1 - 3y$, into the second equation yields the equation $(y + 1)(1 - y) = 0$, and thus the critical points $(-2, 1)$ and $(4, -1)$.

3. (25 points) Write the corresponding linear system of[2]

$$\frac{dx}{dt} = x(2 - x - 3y) \text{ and } \frac{dy}{dt} = y(1 - 2x)$$

at the critical point $(2, 0)$.

Solution: Note that

$$F(x, y) = x(2 - x - 3y) \text{ and } G(x, y) = y(1 - 2x).$$

The Jacobian matrix for the system is

$$\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 2 - 2x - 3y & -3x \\ -2y & 1 - 2x \end{pmatrix}.$$

Thus the corresponding linear system at $(2, 0)$ is

$$\begin{pmatrix} u' \\ w' \end{pmatrix} = \begin{pmatrix} -2 & -6 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix},$$

where $u = x - 2$ and $w = y - 0 = y$.

[2] This system is different than the previous system in Q2.

4. (25 points) Determine whether the critical point $(2, 0)$ of[3]

$$\frac{dx}{dt} = x(2 - x - 3y) \text{ and } \frac{dy}{dt} = y(1 - 2x)$$

is

$\surd$ **asymptotically stable,** ◯ stable (but not asymptotically stable) or ◯ unstable

in the nonlinear system. Justify your answer.

Solution: The eigenvalues of the corresponding linear system are $\lambda = -2,$ and -3. The eigenvalues are real, different and negative. Hence, the equilibrium point $(2, 0)$ is an asymptotically stable node of the nonlinear system.

[3] This is the same system as in Q3.

5. (BONUS optional, 10 points) Give an example of a non-linear autonomous equation for which 0 is a critical point, but not an isolated one.

Solution: